Calcular los siguientes límites:

1 $\lim_{x \to 0} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right)$

2 $\lim_{x \to 0} \frac{\sqrt{x^2+1} - 1}{x^2}$

3 $\lim_{x \to 3} \frac{\sqrt{x^2-5} - 2}{x-3}$

4 $\lim_{x \to \infty} \left(1 - \frac{1}{x^2} \right)^x$

5 $\lim_{x \to 0} \frac{\ln(\cos(3x))}{\ln(\cos(2x))}$

6 $\lim_{x \to \infty} x \cdot \left[\operatorname{arctg}\left(e^x\right) - \frac{\pi}{2} \right]$

7 $\lim_{x \to 0} \left(\operatorname{sen} x \right)^x$

8 $\lim_{x \to +\infty} \left(2^x - 1 \right)^{\frac{2}{x+1}}$

9 $\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \operatorname{sen} x}$

10 $\lim_{x \to 0} \frac{x \cdot \operatorname{sen} x}{1 - \cos x}$

Ejercicios de límites resueltos

Calcular los siguientes límites :

1 $\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)$

$$\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)=\infty-\infty$$

$$\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{x-\ln(1+x)}{x\cdot\ln(1+x)}\right)=\left(\frac{0}{0}\right)=(\text{L`Hôpital})\lim_{x\to 0}\frac{1-\frac{1}{1+x}}{\ln(1+x)+x\cdot\frac{1}{1+x}}=$$

$$\lim_{x\to 0}\frac{\frac{x}{1+x}}{\frac{(1+x)\ln(1+x)+x}{1+x}}=\lim_{x\to 0}\frac{x}{(1+x)\ln(1+x)+x}=\left(\frac{0}{0}\right)=(\text{L`Hôpital})$$

$$\lim_{x\to 0}\frac{1}{\ln(1+x)+(1+x)\cdot\frac{1}{1+x}+1}=\lim_{x\to 0}\frac{1}{\ln(1+x)+2}=\frac{1}{2}$$

2 $\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{x^2}$

$$\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{x^2}=\left(\frac{0}{0}\right)(\text{L`Hôpital})$$

$$\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{x^2}=\lim_{x\to 0}\frac{\frac{x}{\sqrt{x^2+1}}}{2x}=\lim_{x\to 0}\frac{x}{2x\cdot\sqrt{x^2+1}}=\left(\frac{0}{0}\right)=(\text{L`Hôpital})=\lim_{x\to 0}\frac{1}{2\cdot\sqrt{x^2+1}}=\frac{1}{2}$$

3 $\lim_{x\to 3}\frac{\sqrt{x^2-5}-2}{x-3}$

$$\lim_{x\to 3}\frac{\sqrt{x^2-5}-2}{x-3}=\left(\frac{0}{0}\right)=(\text{L`Hôpital})=\lim_{x\to 3}\frac{\frac{x}{\sqrt{x^2-5}}}{1}=\lim_{x\to 3}\frac{x}{\sqrt{x^2-5}}=\frac{3}{2}$$

4 $\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)^x$

$$\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)^x=1^\infty$$

Intentamos conseguir $\left(1+\frac{1}{n}\right)^n=e$, más sencillo que hacerlo por L`Hôpital.

$$\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)^x=\lim_{x\to\infty}\left(1+\frac{1}{-x^2}\right)^{x\left(\frac{-x}{-x}\right)}=\lim_{x\to\infty}\left(1+\frac{1}{-x^2}\right)^{-x^2\cdot\frac{1}{x}}=e^{\lim_{x\to\infty}\frac{-1}{x}}=e^0=1$$

8 $\lim\limits_{x\to+\infty}\left(2^x-1\right)^{\frac{2}{x+1}}$

$\lim\limits_{x\to+\infty}\left(2^x-1\right)^{\frac{2}{x+1}}=\infty^0$

$A=\lim\limits_{x\to+\infty}\left(2^x-1\right)^{\frac{2}{x+1}} \quad \Rightarrow \ \ln A=\lim\limits_{x\to+\infty}\frac{2}{x+1}\ln\left(2^x-1\right)=\left(\frac{\infty}{\infty}\right)=$ (L´Hôpital)=

$\lim\limits_{x\to+\infty}\frac{2\ln\left(2^x-1\right)}{x+1}=\lim\limits_{x\to+\infty}\frac{2\cdot\frac{2^x\ln 2}{2^x-1}}{1}=\lim\limits_{x\to+\infty}\frac{2\cdot 2^x\ln 2}{2^x-1}=\left(\frac{\infty}{\infty}\right)=$ (L´Hôpital)=

$\lim\limits_{x\to+\infty}\frac{2\cdot 2^x(\ln 2)^2}{2^x\ln 2}=\lim\limits_{x\to+\infty}2\cdot\ln 2=\ln 4$

$\ln A=\ln 4\Rightarrow A=4$ Solución: $\lim\limits_{x\to+\infty}\left(2^x-1\right)^{\frac{2}{x+1}}=4$

9 $\lim\limits_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\operatorname{sen}x}$

$\lim\limits_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\operatorname{sen}x}=\left(\frac{0}{0}\right)=$ (L´Hôpital)=

$\lim\limits_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}=\left(\frac{0}{0}\right)=$ (L´Hôpital)= $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\operatorname{sen}x}=\left(\frac{0}{0}\right)=$ (L´Hôpital)

$\lim\limits_{x\to 0}\frac{e^x+e^{-x}}{\cos x}=\frac{2}{1}=2$

10 $\lim\limits_{x\to 0}\frac{x\cdot\operatorname{sen}x}{1-\cos x}$

$\lim\limits_{x\to 0}\frac{x\cdot\operatorname{sen}x}{1-\cos x}=\left(\frac{0}{0}\right)=$ (L´Hôpital)= $\lim\limits_{x\to 0}\frac{\operatorname{sen}x+x\cdot\cos x}{\operatorname{sen}x}\left(\frac{0}{0}\right)=$ (L´Hôpital)=

$\lim\limits_{x\to 0}\frac{\cos x+\cos x+(-\operatorname{sen}x).x}{\cos x}=\frac{2}{1}=2$